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LETTER TO THE EDITOR

Discussion of "Elastic anisotropy of short-fibre reinforced composites", *Int. J. Solids Structures*, Vol. 29, No. 23, pp. 2933–2944 (1992)

In the subject paper, Sayers employs an ingenious approximate scheme for estimating the effective elastic response of biphase short-fiber composites with orthotropic orientation distributions. The purpose of this discussion is threefold: To analyse Sayers' method in the light of recent developments in the theory of homogenization of such composites; to propose some experimental research directions, associated with these developments; and to point out additional relevant references for the problem.

To facilitate the discussion, it is recalled that, for a biphase composite, consisting of a matrix of stiffness $\mathbf{C}^1$, containing a volumetric concentration v_2 of inhomogeneities, or fibers, of stiffness $\mathbf{C}^2$, the effective stiffness may be expressed as

$$\mathbf{C}^* = \mathbf{C}^1 + v_2\langle(\mathbf{C}^2 - \mathbf{C}^1)\mathbf{A}\rangle, \tag{1}$$

where the (average) strain concentration tensor $\mathbf{A}$ was introduced, following Hill (1963). This tensor is generally unknown, and it is the approximation chosen for $\mathbf{A}$ that characterizes the different homogenization methods. Thus, Voigt's method (1928) corresponds to unitary $\mathbf{A}$, and the Mori-Tanaka (1973) scheme has

$$\mathbf{A} = \mathbf{T}[(1-v_2)\mathbf{I} + v_2\langle\mathbf{T}\rangle]^{-1}, \tag{2}$$

where $\mathbf{T}$ is defined in the subject paper, on p. 2937.

In eqns (2)–(3), angled brackets denote orientational averaging, weighted by a fiber orientation probability density function. Simple misorientations have been studied with this approach by Takao *et al.* (1982) and Zhao *et al.* (1989), in conjunction with the Mori-Tanaka assumption (2). Ferrari and Johnson (1989) presented the general harmonic analysis methods for the treatment of the arbitrary orientation distribution function, and discussed its application to the Mori-Tanaka formalism. On these foundations, and the matricial reformulation method (Ferrari and Marzari, 1989), extensive sensitivity studies have been performed (Marzari and Ferrari, 1992) to investigate the dependence of the composite moduli on the fiber geometry, concentration, constitution and orientation distribution for arbitrarily textured composites.

The problem of the homogenization of a textured composite thus consists of two parts: The identification of an appropriate concentrator $\mathbf{A}$ and the development of a weighted orientational averaging procedure. On the latter, Sayers does not introduce any novelty, while his choice of concentrator is innovative: He chooses $\mathbf{A}$ to correspond to the Mori-Tanaka concentrator (2) *for the case of perfect alignment*, i.e. with no angled brackets about $\mathbf{T}$. This apparently minor change is in reality of momentous consequence, well beyond the associated computational simplifications.

To elaborate: The admissibility of a particular concentrator, and thus of the associated homogenization scheme, depends on the texture of the composite to be homogenized. In particular, the Mori-Tanaka method (2) (i) yields an asymmetric effective stiffness tensor, (ii) violates the variational bounds on the effective modulus, and (iii) fails to recover the fiber moduli at the unitary concentration limit, when applied to the arbitrarily textured composite (Ferrari, 1991), and is thus inadmissible in this context. For macroscopically isotropic composites, however, the Mori-Tanaka theory is fully admissible, under criteria (i)–(iii), unless the fibers consist of anisotropic material (*ibidem*). The admissibility of the general concentrator $\mathbf{A}$ has been studied, essentially in terms of the stated criteria (Ferrari, 1992), and none of the currently applied homogenization schemes has been found to comply with these criteria. In the same work, the entire family of concentrators of the type

$$\mathbf{A} = \{\mathbf{I}+[f(v_2)\mathbf{E}+g(v_2)\mathbf{I}]\mathbf{S}^1(\mathbf{C}^2-\mathbf{C}^1)\}^{-1} \tag{3}$$

was proven to give a symmetric $\mathbf{C}^*$ for any textured biphase composite. In (3), $\mathbf{S}^1$ is the matrix compliance tensor, while $f(\cdot)$ and $g(\cdot)$ are arbitrary functions of the fiber concentration. Compliance with the admissibility conditions at the zero and unitary concentration limits is guaranteed, by imposing that $f(0) = 1$, $f(1) = g(0) = g(1) = 0$.

Thus, the simplest fully admissible concentrator corresponds to $f(x) = 1-x$, $g(x) \equiv 0$, provided the associated moduli comply with the known variational bounds. Much is to be said, concerning this concentrator:

(1) It recovers the exact effective bulk modulus for the composite sphere assemblage of Hashin (1962), as shown in Ferrari (1993).

(2) It recovers the exact moduli for the case of composites with equal shear or bulk moduli in fiber and matrix (Ferrari *et al.*, 1993).

(3) It is consistent with the most reliable experimental data on isotropic and aligned composites (*ibidem*).

(4) It coincides with Sayers' approximation, as may be shown from the previous formulae, through a few lines of tensor manipulations. As Sayers points out, this "approximation" may thus be interpreted as first homogenizing in each direction, under the assumption that all fibers are consistently aligned, and then performing a texture-weighted orientational averaging. In light of remarks (1)–(3), Sayers' approximation is concluded to be an extremely valid one, and possibly not an approximation at all.

(5) That this assumption be the more valid the less the composite is unidirectionally aligned is beyond doubt. However, in the particular context of injection molded composites, in which Sayers casts his discussion, the effect of the fiber dimensional distribution is comparable with the misalignment effects (Boscolo *et al.*, 1991)—both effects being small, as far as the longitudinal moduli are concerned (*ibidem*).

Having discussed the relation between Sayers' hypotheses and the admissibility criteria of effective medium theory, this note concludes with a few additional comments.

First, concerning his Fig. 2, it is pointed out that while there is a maximum achievable second-phase concentration for a mono-size fiber distribution, no such limit may be established for composites with a fiber size gradation. Furthermore, the Mori-Tanaka theory, employed for the predictions represented in this Figure, is scale-invariant, as it expresses the geometric dependance through Eshelby's tensor, that is a function of the fiber aspect ratios only. Thus, the prediction of the theory at high concentrations must be considered as a valid probing ground for the admissibility of the theory itself.

Second, Sayers proposes that the measurement of the six ultrasonic velocities expressed in his eqns (38)–(43) be used to determine the *five* texture coefficients, that fully determine the elastic response, at least in the case of orthotropy macroscopic and in the presence of an isotropic matrix. This inversion procedure appears to be ill-posed, (and much more so for transverse isotropy). It is proposed that the procedure be supplemented with a sixth unknown, corresponding to the value of $f(v_2)$ at the known fiber concentration v_2 at which the characterization experiments are performed. It is noted that this value enters in Sayers' anisotropy factors a_i, when calculated under a homogenizing assumption of the type (3) [with say $g(\cdot) \equiv 0$]. The advantage in this procedure is obviously that the elastically relevant texture is determined *simultaneously with the homogenization scheme*. This removes the uncertainty due to the use of the approximations inherent in the homogenizing criteria.

Finally, Sayers' discussion is limited to the texture coefficients of rank two, and points to the possibility of employing the coefficients determined from the inversion procedure described above for the prediction of the strength of the composite, according to the approach of Templeton (quoted in the subject paper). In this context, a more detailed discussion of the relevance of the higher order coefficients would be welcome, especially in view of the fact that, while texture coefficients of rank higher than four do not effect the macroscopic elastic properties (Ferrari and Johnson, 1989), at least for isotropic-matrix composites, it is not at all clear that the same should be true for strength. In this sense, the few coefficients obtained from ultrasonic experiments may be insufficient.

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